Chapter 14.1 – 14.3

Random Variables and Probability Models

D. Raffle
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Overview

In Chapters 12–13, we saw:

- How to decribe random phenomena as trials
- How to use probability rules to describe the outcomes of these trials

In this chapter, we will:

- Describe the long-term behavior of random phenomena using **probability distributions**
- See how many phenomena can be modeled using several common named distributions

Background: Insurance

Say a particular insurance company offers a "death and disability" policy with the following payouts:

- $10,000 for death
- $5,000 for disability

They charge a $50 premium per year. How profitable do they think they'll be?

- We need to know the probability that a given client will be killed or disabled
- With this information, we can find the **expected value** of their product
- We can also find the **standard deviation**, which can tell us about the uncertainty they'll face

Probability Models

Recall that a mathematical model is just a formula that is used to represent the real world

- $Area = Base \times Height$
- $Speed = \frac{distance}{time}$

A statistical model is a mathematical model which accounts for the uncertainty of random events.

- $\hat{y} = b_0 + b_1 x$

A **probability distribution** is a specific type of statistical model which describes the probability of certain events happening.

- $X \sim N(\mu,\ \sigma)$

Definitions

A **Random Variable**:

- A variable whose value is random.
- Each time we observe the variable, it is a random trial.
- A particular coin flip is a random trial, but flipping a coin is a random variable
- We typically denote random variables with capital letters, like X and its value using lower-case letters
- "$X = x$" means "the variable X taking the value x"
- $P(X = x)$ is the probability that X takes the value x
- We usually use the shorthand $P(x)$

Insurance: The Variable

For our insurance example, we'll denote the amount paid out X. Suppose that the following is true:

- One in one thousand people will be killed in a given year, on average.
- Two in one thousand people will be disabled, on average.

How do we represent this? Usually with a table

Outcome	Payout (x)	Probability $P(x)$
Death	$10,000	$\frac{1}{1000}$
Disability	$5,000	$\frac{2}{1000}$
Neither	$0	$\frac{997}{1000}$

Discrete vs. Continuous

A **discrete** random variable:

- A variable whose outcomes we can list
- The payout is discrete, because we can list all three outcomes

A **continuous** random variable:

- There are too many possibilities to list
- We usually deal with ranges
- Usually measurements are continuous
- Something can weigh 1 kg, 1.1 kg, 1.11 kg, 1.111 kg, etc.
- Technically, there are an **infinite** number of possible outcomes

Valid Distributions

The outcomes in a probability distribution make up the **sample space**, so we need to follow the same rules as in Chapter 12

In the discrete case:

- All of the probabilities need to add up to exactly 1
- Outcomes cannot overlap
- Every probability needs to ≥ 0

In the continuous case:

- The same basic rules apply, but we need calculus to verify them
- For this class, just trust that they're valid

14.1 The Expected Value

The **expected value** is the **long-term average outcome** or **population mean** of a random variable.

- If we repeatedly observe it, what's the average?
- If we haven't observed a particular trial yet, what do we expect to happen?

Notation

- $E(X)$ or μ
- We use $E(X)$ if we're just describing the variable
- We use μ if it's the parameter of a model, e.g. in the Normal Distribution

Calculation

- $E(X) = \sum xP(x)$

Insurance: The Expected Value

Outcome	Payout (x)	Probability $P(x)$
Death	$10,000	$\frac{1}{1000}$
Disability	$5,000	$\frac{2}{1000}$
Neither	$0	$\frac{997}{1000}$

Finding $E(X)$

- $E(X) = \sum xP(x)$
- $E(X) = \$10000\left(\frac{1}{1000}\right) + \$5000\left(\frac{2}{1000}\right) + \$0\left(\frac{997}{1000}\right)$
- $E(X) = \frac{\$10000}{1000} + \frac{\$10000}{1000} = \$10 + \$10 = \$20$

Insurace: Interpreting $E(X)$

We found $E(X) = \$20$. What does this tell us?

- For a given customer, the company expects to spend $20
- Remember that they charge $50 for the policy
- For each customer, we have an expected profit of $30

Note

- They only ever pay out $10,000, $5,000 or $0
- They'll either lose $9,950 or $4950, or they can keep all $50
- Since most people will not get injured or killed, the larger number of customers who give them pure profit balance out those who cost them thousands
- This is the basis for all insurance/warranty plans

14.2 The Standard Deviation

For the insurance example, we had a wide range of outcomes. This means that there's a large amount of **uncertainty** or **variability** from customer to customer.

- Just like with the spread of samples, we describe the spread of probability distributions with the **standard deviation**.
- In samples, we found the variance as the *average squared distance* from observations to the mean
- The standard deviation was the square root of the variance
- For distributions, we use the *expected squared distance* from each outcome to the *expected value*

The Standard Deviation

Notation:

- We use $VAR(X)$ or σ^2 to denote the distribution's variance
- We use $SD(X)$ or σ to denote the distribution's standard devation

Calculation:

- $VAR(X) = \sigma^2 = \sum (x - \mu)^2 P(x) = \sum x^2 P(x) - \mu^2$
- $SD(X) = \sqrt{VAR(X)}$ or $\sigma = \sqrt{\sigma^2}$

Insurance: The Standard Deviation

For our insurance company, $X = \{\$10000,\ \$5000,\ \$0\}$and $E(X) = \$20$.
Finding the standard deviation:

- $\sigma^2 = \sum(x-\mu)^2 P(X)$
- $\sigma^2 = (10000-20)^2\left(\frac{1}{1000}\right) + (5000-20)^2\left(\frac{2}{1000}\right) + (0-20)^2\left(\frac{997}{1000}\right)$
- $\sigma^2 = (9980)^2\left(\frac{1}{1000}\right) + (4980)^2\left(\frac{2}{1000}\right) + (-20)^2\left(\frac{997}{1000}\right)$
- $\sigma^2 = (9960040)\left(\frac{1}{1000}\right) + (24800400)\left(\frac{2}{1000}\right) + (400)\left(\frac{997}{1000}\right)$
- $\sigma^2 = 99600.4 + 49600.8 + 398.8$
- $\sigma^2 = 149600$
- $\sigma = \sqrt{149600} = 386.7816$

Insurance: The Standard Deviation

What does it tell us that $\sigma = \$386.78$?

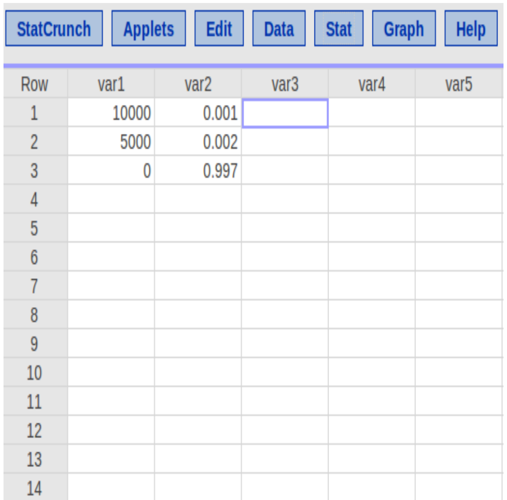
- There's a big difference between paying out thousands or pocketing $50
- While we expect to make $30 per person in the long term, there is a lot of uncertainty about individual customers
- They'll probably make a lot of profit if they insure thousands, but insuring a small number of people is a lot of risk

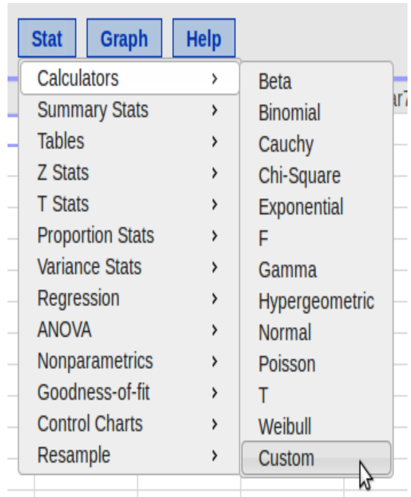
In StatCrunch

This isn't an algebra class, so we can use StatCrunch to do the heavy lifting.

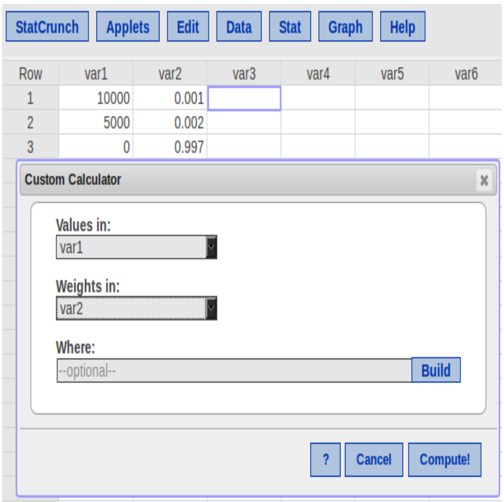
- Open a blank data set
- Enter the values of X as on column
- Enter the probabilities as another column
- `Stat` $\rightarrow$ `Calculators` $\rightarrow$ `Custom`
- Select the columns
- Hit `Compute`

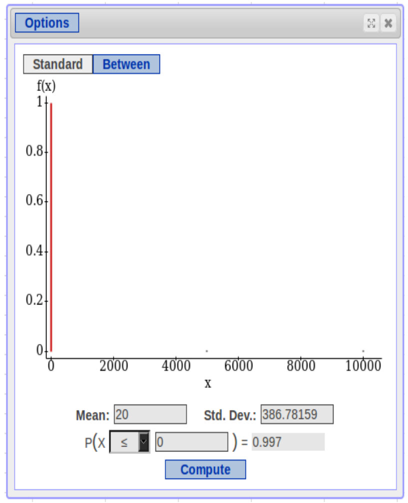
In StatCrunch





In StatCrunch





Example: Refurbished Computers

Say a company sells custom computers to businesses. A particular client orders two new computers, but they refuse to take refurbished (repaired) computers.

- Someone made an error in stocking, and of the 15 computers in the stock room 4 were refurbished
- Since the company filled the order by randomly picking up computers from the stock room, there was a chance they shipped 0, 1, or 2 refurbished computers
- If the client just gets one refurbished computer, they'll ship it back at the computer company's expense, costing them $100
- If they get two refurbished machines, they'll cancel the entire order, and the computer comany will lose $1000.

Refurbished Computers: Probabilities

Getting Two New Computers

- There are 11 new machines out of 15
- $P(\text{first new}) = \frac{11}{15}$
- $P(\text{second new} \mid \text{first new}) = \frac{10}{14}$
- $P(\text{both new}) = P(\text{first new AND second new} \mid \text{first new})$
- $P(\text{both new}) = P(\text{first new}) \times P(\text{second new} \mid \text{first new})$
- $P(\text{both new}) = \frac{11}{15} \times \frac{10}{14} = \frac{110}{210} \approx 0.524$
- There's a 52.4% chance the computer company doesn't lose money

Refurbished Computers: Probabilities

Getting Two Refurbished Computers

- There are 4 refurbished machines out of 15
- $P(\text{first refurb.}) = \frac{4}{15}$
- $P(\text{second refurb.} \mid \text{first refurb.}) = \frac{3}{14}$
- $P(\text{both refurb.}) = P(\text{first refurb. AND second refurb.} \mid \text{first refurb.})$
- $P(\text{both refurb.}) = P(\text{first refub.}) \times P(\text{second refub.} \mid \text{first refurb.})$
- $P(\text{both refurb.}) = \frac{4}{15} \times \frac{3}{14} = \frac{12}{210} \approx 0.057$
- There's a 5.7% chance the computer company loses $1000

Refurbished Computers: Probabilities

Getting One Refurbished Computer

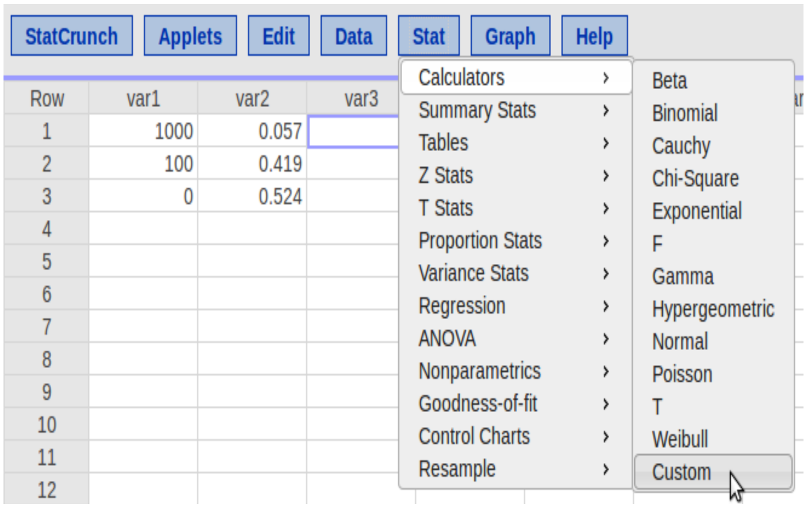
- Note that the sample space only includes getting two new computers, getting two refurbished computers, or one of each
- $P(\text{both new}) + P(\text{both refurb}) + P(\text{one new}) = 1$
- $P(\text{one new}) = 1 - (P(\text{both new}) + P(\text{both refurb}))$
- $P(\text{one new}) = 1 - (0.524 + 0.057)$
- $P(\text{one new}) = 1 - 0.581$
- $P(\text{one new}) = 0.419$
- There's a 41.9% chance the computer company loses $100

Refurbished Computers: The Probability Distribution

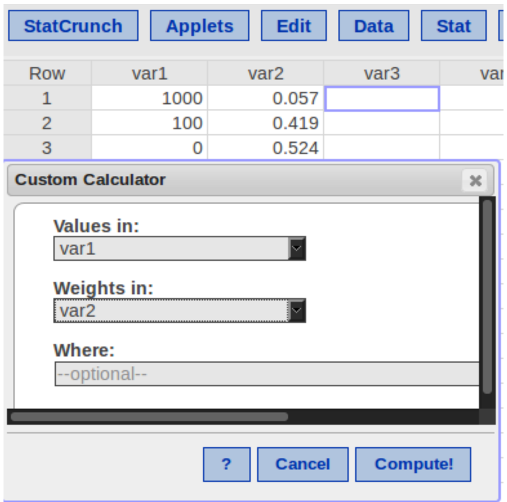
Outcome	Money Lost (X)	Probability $P(x)$
Both Refurbished	$1000	0.057
One New	$100	0.419
Both New	$0	0.524

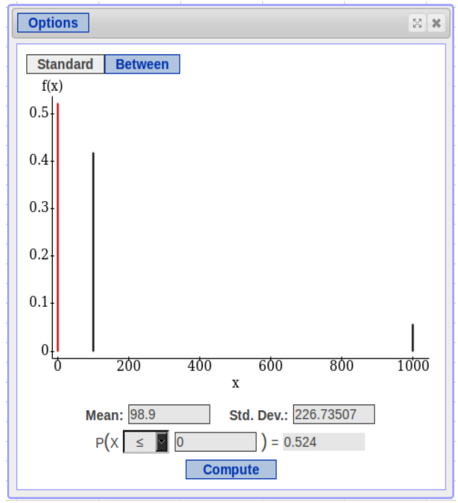
Now we can use StatCrunch to find $E(X)$ and $SD(X)$

In StatCrunch



In StatCrunch





Refurbished Computers: Interpretation

$E(X) = \$98.9$

- The company should expect to lose $98.90 for this mistake
- While the most likely outcome is the company not losing money, the rare event of them losing $1000 carries a lot of weight

$SD(X) = \$226.74$

- There's a lot of uncertainty in this scenario
- They could lose 0, $100, or $1000. Since there's only two computers, it's hard to predict exactly what will happen this one time.
- If this mistake were to be repeated, the outcome might be completely different

14.3 Combining Random Variables

Let's head back to the insurance example.

- We looked at the company's expected payout for a single person
- What if the company lowered the price of the premium from $50 to $45?
- What if we doubled the payouts?
- What would the expected payout be for two people? The standard deviation?

It turns out we have simple rules for these problems.

Adding a Constant

We saw in earlier chapters that adding or sutracting a constant from each value in a sample shifts the mean, but leaves the measures of spread alone. The same is true for random variables.

- $E(X \pm c) = E(X) \pm c$
- $VAR(X \pm c) = VAR(X)$
- $SD(X \pm c) = SD(X)$

What if our insurance company lowered the premium by $5?

- $E(X) = \$20$, so the expected profit was $50 - $20 = $30
- If we lose an additional $5 from each customer, the expected profit is $45 - $20 = $25
- The standard deviation will stay the same

Multiplying by a Constant

In earlier chapters, we saw that the mean and standard deviation were both scaled when multiplying by a constant. The same holds true for random variables.

- $E(aX) = aE(X)$
- $VAR(aX) = a^2 VAR(X)$, because we **square** all the differences from the means
- $SD(aX) = |a| SD(X)$, again because we are **squaring** things then taking the square root

So what happens if we double all payouts for the insurance company?

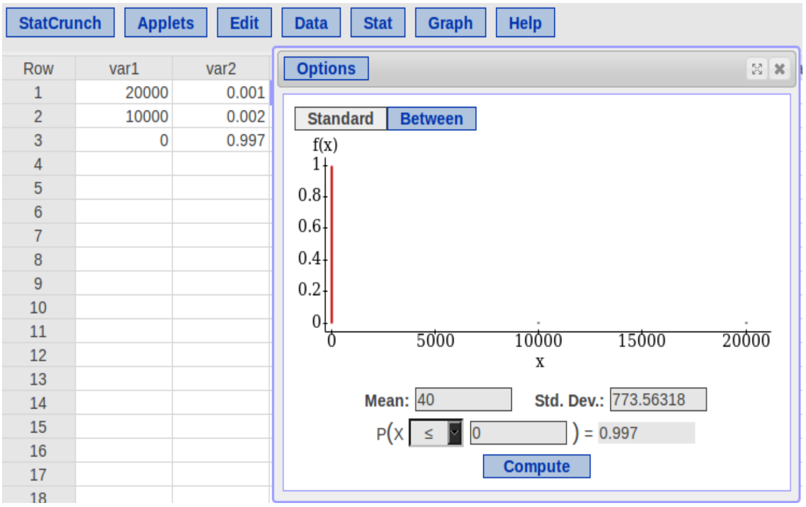
Doubling Payouts

Outcome	Payout (x)	Probability $P(x)$
Death	$20,000	$\frac{1}{1000}$
Disability	$10,000	$\frac{2}{1000}$
Neither	$0	$\frac{997}{1000}$

Using StatCrunch,

- $E(X) = \$40$
- $SD(X) = \$773.56$

Doubling Payments



What Happened?

$E(X) = \$40$

- Because we doubled all payouts, the expected payout has doubled

$SD(X) = \$773.56 = 2 \times \$386.78$

- By doubling all payouts, we doubled the differences between the outcomes
- Since the outcomes are further apart, we doubled the range of outcomes
- For any one customer, there is much more uncertainty in how much the company will lose

Adding Variables

Instead of looking at a single customer, we'll look at two. Call them Mr. X. and Mrs. Y.

Isn't looking at two customers the same as multiplying one customer's payouts by two?

- Not quite.
- Mr. X might die, but Mrs. Y survives the year unharmed
- Mr. X could stay safe, while Mrs. Y gets maimed
- We have different rules for adding random variables

Adding Variables

If X and Y are independent,

- $E(X+Y) = E(X) + E(Y)$
- $VAR(X+Y) = VAR(X) + VAR(Y)$
- $SD(X+Y) = \sqrt{VAR(X+Y)}$

Insurance: Adding Variables

So what should the insurance company expect with Mr. X and Mrs. Y?

$E(X+Y)$:

- $E(X+Y)=E(X)+E(Y)=20+20=40$

$SD(X+Y)$

- $VAR(X+Y)=VAR(X)+VAR(Y)$
- $VAR(X+Y)=149600+149600=299200$
- $SD(X+Y)=\sqrt{299200}=\$546.99$

Insurance: Adding Variables

What happened?

- By doubling the number of policies, we've doubled the expected payout (but also the premiums we collect)
- Notice that $SD(X+Y) < SD(2X)$
- By insuring multiple people, we spread the risk around between the customers
- Even though the expected payout is the same as offering one policy with twice the coverage, we've reduced the uncertainty involved
- It's the same profit with less uncertainty

Subtracting Variables

Instead of adding variables, we can also subtract them. In general,

- $E(X \pm Y) = E(X) \pm E(Y)$
- $VAR(X \pm Y) = VAR(X) + VAR(Y)$
- $SD(X \pm Y) = \sqrt{VAR(X) + VAR(Y)}$

Note that, even when we subtract the variables, we **always** add the variances

$X + X \neq 2X$

Like we saw with the insurance example, adding two random variables with the same distribution is not the same as multiplying one of them by two

- For the insurance, $2X = \{\$0, \$10000, \$20000\}$
- For both customers, the sample space includes all possible combinations of X added together

We need to be careful with notation

- For a small number of variables, we might use X, Y, and Z
- For more variables, we often number them $X_1, X_2, \ldots, X_n$ where n is our number of variables

Multiple Observations

When we observe the same variable multiple times, like having two insurance customers, we can simplify things. For each observation, the mean and variances are the same, so:

- $E(X_1 + X_2) = E(X_1) + E(X_2) = 2 \times E(X)$
- $E(X_1 + X_2 + \ldots + X_n) = n \times E(X)$
- $VAR(X_1 + X_2) = VAR(X_1) + VAR(X_2) = 2 \times VAR(X)$
- $VAR(X_1 + X_2 + \ldots + X_n) = n \times VAR(X)$
- $SD(X_1 + X_2 + \ldots + X_n) = \sqrt{VAR(X_1 + X_2 + \ldots + X_n)}$

Example: Week and Weekend Shifts

Say you're a waiter at a restauarant and a large portion of your income is in tips. During a typical 5-day work week, you make an average of $1200 with a standard deviation of $150. On the weekends, you average $400 in tips with a standard deviation of $70. Let X represent the 5-day work week and Y represent the weekend.

What do you expect to make for an entire 7-day week?

- $E(X+Y) = E(X) + E(Y) = \$1200 + \$400 = \$1600$

What is the standard deviation for the entire week?

- $VAR(X+Y) = VAR(X) + VAR(Y) = 150^2 + 70^2$
- $VAR(X+Y) = 22500 + 4900 = 27400$
- $SD(X+Y) = \sqrt{X+Y} = \sqrt{27400} \approx \$165.53$

Example: Week and Weekend Shifts

Say that you typically make within one standard deviation of the mean. What's a typical weekly salary for you?

- $E(X+Y) - SD(X+Y) = 1600 - 165.53 = \$1434.47$
- $E(X+Y) + SD(X+Y) = 1600 + 165.53 = \$1765.53$
- You typically make between $1434.47 and 1765.53

Only on really good weeks do you make more than two standard deviations above the mean. How much do you make in a really good week?

- $E(X+Y) + 2 \times SD(X+Y) = 1600 + 2 \times 165.53 = 1600 + 331.06$
- $E(X+Y) + 2 \times SD(X+Y) = \$1931.06$
- Having a really good week means earning at least $1931.06

Example: Monthly Income

Using the same information from before, let's call weekly income W.
$E(W) = \$1600$, $SD(W) = \$165.53$

How much would you expect to make in a month?

- $E(M) = E(W_1 + W_2 + W_3 + W_4) = 4 \times E(W)$
- $E(M) = 4 \times 1600 = \$6400$

What is the standard deviation for a month?

- $VAR(M) = VAR(W_1 + W_2 + W_3 + W_4) = 4 \times VAR(W)$
- $VAR(M) = 4 \times 165.53^2 = 4 \times 27400.18 = 109600.7$
- $SD(M) = \sqrt{VAR(M)} = \sqrt{109600.7} = 331.06$

Summary

- A **random variable** is a variable with a random outcome
- We describe the behavior of a random variable with a **probability distribution**
- The **expected value** or **mean** of a random variable describe the long-term average
- The **variance** and **standard deviation** describe the uncertainty or spread of a distribution
- When we add random variables, we can add the expected values and variances (but not the standard deviation)